

Topological Labeling of Periodic Graphs

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Abstract

In this paper, we extend the definition of topological labeling of finite directed acyclic graphs to infinite acyclic periodic graphs. Unlike finite acyclic graphs, we show that acyclic periodic graphs cannot be topologically labelled with integers in general. However, we show that these graphs can be labelled with rationals. We also give sufficient conditions for the existence of an integral topological labeling. The labels are obtained by solving a sequence of linear programs.

1 Introduction

In this paper we will consider the problem of creating a topological labeling of the arcs of acyclic periodic graphs. This work extends the concept of topological labeling of finite acyclic graphs. Periodic graphs are infinite graphs that have a periodic structure. Various aspects of infinite periodic graphs have been studied by different authors [5], [8], [4], [6], [1], [7] and [3]. Periodic graphs arise naturally when modeling systems of uniform recurrence equations. Periodic graphs can be used to model infinite horizon production problems and the parallelization of DO loops. Karp *et.al.* [5] studied the related multidimensional problem of finding cycles in graphs with multidimensional arc weights. Periodic graphs with 1-dimensional weights were studied in detail by Orlin [8]. Iwano and Steiglitz [4] used a 2-dimensional dynamic periodic graph to model VLSI circuits and also raised the question of finding cycles in these graphs [4]. Kosaraju and Sullivan [6], developed an $O(n \log n Z)$ algorithm (Z is the time complexity of solving a linear program) to solve this cycle recognition problem. Cohen and Megiddo [1] established that for fixed dimension, the cycle recognition problem was in NC. They gave a strongly polynomial algorithm for the fixed dimension case. Hötfing and Wanke [3] developed algorithms for determining minimum cost paths in periodic graphs.

In this paper, we consider the problem of topologically labelling the nodes of an acyclic periodic graph. This is a natural generalization of the integral topological labelling of finite acyclic graphs. We first review the concept of topological labeling of finite acyclic graphs

and review the algorithm for labeling the nodes of a finite acyclic graph topologically. In Section 2, we develop an algorithm for topologically labeling the nodes of an acyclic periodic graph. In Section 3, we give sufficient conditions for an acyclic periodic graph to have an integral lexicographic labeling, and we also provide a necessary condition.

Definition 1 *Let $G = (V, E)$ be a finite acyclic graph. A topological labeling of the nodes of G is a mapping $l : V \rightarrow \{1, 2, \dots, n\}$ such that whenever there is an arc (v, w) , $l(v) < l(w)$.*

The next two lemmas are well known (see Even [2]). We state them without proof.

Lemma 1 *Let $G = (V, E)$ be a finite graph. Then G is acyclic if and only if there exists a topological labeling of the nodes of G .*

Algorithms for topologically labeling the nodes of an acyclic graph use the following lemma to generate the topological labeling of the nodes of the graph.

Lemma 2 *Let $G = (V, E)$ be an acyclic graph. There is at least one node in G with indegree zero.*

An algorithm for generating a topological labeling of the nodes of an acyclic graph is the following:

```

begin
   $G' = G$ ;
  for  $k = 1, n$  do
    begin
      select  $i \in G'$  with indegree 0;
      delete  $i$  from  $G'$ ;
       $l(i) = k$ ;
    end
  end

```

At the end, the algorithm assigns a unique label in $\{1, 2, \dots, n\}$ to each node $i \in V$, and the result is a topological labeling of the nodes in G . The algorithm can be implemented to run in $O(m)$ time.

2 Lexicographically Labeling an Acyclic Period Graph

We first define the concept of a periodic graph. Periodic graphs are infinite graphs that have a periodic structure. More formally, we define a periodic graph as per Cohen and Megiddo [?].

Definition 2 Given a generator graph $G = (V, E, C)$, where $C : E \rightarrow Z^d$ is a weight function on the arcs of G , the dynamic periodic graph $G^\infty = (V^\infty, E^\infty)$ defined by G has:

$$V^\infty = \{v^z \mid z \in Z^d, v \in V\}$$

$$E^\infty = \{(u^z, v^{z+C_{uv}}) \mid z \in Z^d (u, v) \in E\}.$$

We refer to C_{uv} as the weight of arc (u, v) . We refer to d as the dimension of the periodic graph. Figure 1 illustrates a 2 dimensional generator graph and the corresponding periodic graph is given in Figure 2. We first note that Lemma 2 does not extend to acyclic periodic graphs. For example consider the periodic graph shown in Figure 2. It can be seen easily that the periodic graph is acyclic. However, no node has indegree zero. One might conjecture that topological labeling can be extended directly to infinite graphs; however one cannot assign an integer label $l(v)$ to each node v , such that for each $(i, j) \in E$, $l(i) < l(j)$.

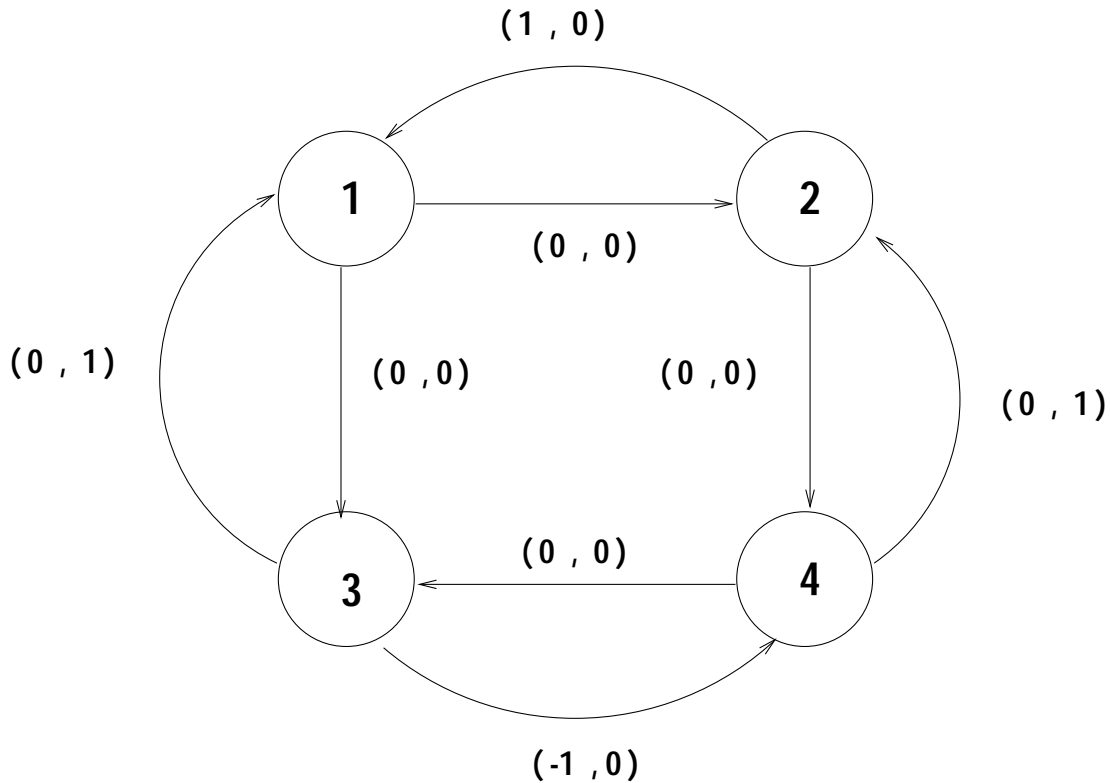


Figure 1: Two Dimensional Generator Graph

We show via a contradiction that no integral topological labeling exists. Consider two nodes k and m shown in Figure 2. Let $l(k)$ and $l(m)$ denote the integral topological labels of k and m respectively. For the topological labeling to be valid, the length of any path between k and m is bounded by $l(m) - l(k)$. Note that there paths of arbitrary lengths

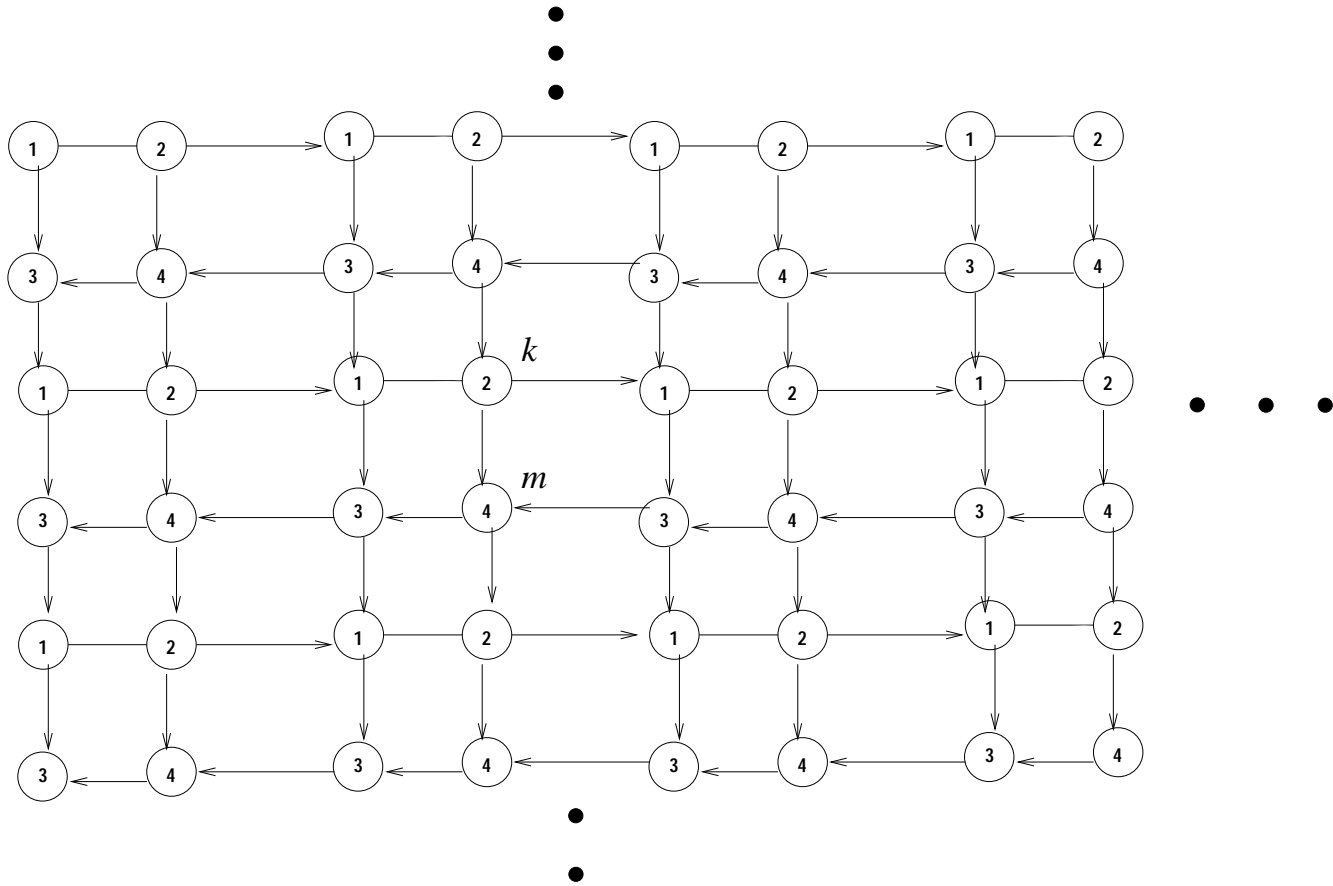


Figure 2: Two Dimensional Periodic Graph

between nodes k and m . Therefore the no integral topological labeling is possible in this case.

We give the following definitions that will be used in the development of a topological labeling scheme for periodic graphs.

Definition 3 A non-zero vector $v \in R^n$ is said to be lexicographically positive if its first non-zero component is greater than zero. If $v = 0$ then it is lexicographically zero. The vector v is lexicographically negative if $-v$ is lexicographically positive.

Definition 4 A vector $v \in R^n$ is said to be lexicographically greater than $w \in R^n$ if $v - w$ is lexicographically positive. This is represented as $v \succ w$. We define lexicographically less than (\prec) and lexicographically equal in the same manner. We say that v is lexicographically greater than or equal to w (represented as \succeq) if $v - w$ is lexicographically greater than zero or equal to zero. We define lexicographically less than or equal to (\preceq) similarly.

Definition 5 A lexicographic labeling of a graph (infinite or not) is an assignment of vectors $A()$ to each node of the graph. A lexicographic labeling is a strong topological labeling if for each arc (v, w) , $A(v) \prec A(w)$. It is weakly topological if for each arc (v, w) , $A(v) \preceq A(w)$.

Definition 6 Let $A(j)$ be the label of node $j \in V$. An arc (v, w) in an infinite graph is weakly lexicographically labeled or A -weak if $A(v) \preceq A(w)$ and strongly lexicographically labeled or A -strong if $A(v) \prec A(w)$.

Definition 7 Let A be a vector labeling of the nodes in a graph G . Let d be the dimension of A . A vector labeling A' is defined as a refinement of A if the dimension of A' is $d + 1$ and $A'(j) = A(j)$ for the first d dimensions for all nodes j .

We now state the main theorem of this paper.

Theorem 3 The following statements are equivalent:

1. A periodic graph is acyclic.
2. A periodic graph has an integral lexicographic topological labeling.
3. A periodic graph has a rational topological labeling.

We prove this theorem by showing that condition 3 implies condition 1, condition 2 implies condition 3 and then condition 1 implies condition 3.

In the next Lemma we show that condition 3 implies condition 1.

Lemma 4 Let G be a finite or an infinite graph. If G has a rational topological labeling then G is acyclic.

Proof:

Let $v_1, v_2, \dots, v_t, v_1$ be a cycle in G . Let $L(v_i)$ be a rational topological label of node v_i . Then

$$L(v_1) < L(v_2) < L(v_3) < \dots < L(v_t) < L(v_1),$$

a contradiction. □

We now show that condition 2 implies condition 3.

Lemma 5 If an infinite graph has a d -dimensional lexicographic labeling of which the first $d - 1$ dimensions are integral, then the infinite graph has a rational topological labeling.

Proof:

We prove this theorem via a construction. If $d \geq 2$, we transform the d dimensional lexicographic labeling into a $d - 1$ dimensional lexicographic labeling such that the first $d - 2$ dimensions are integral. (The dimension of a labeling is the number of components

in each label.) Let $A_i(v)$ denote the i -th component of the vector associated with node v . For each node v such that $A_d(v) \geq 0$ replace $A_{d-1}(v)$ by

$$A'_{d-1}(v) = A_{d-1}(v) + \frac{A_d(v)}{1 + 2A_d(v)}.$$

For each node v such that $A_d(v) < 0$ replace $A_{d-1}(v)$ by

$$A'_{d-1}(v) = A_{d-1}(v) + \frac{A_d(v)}{1 - 2A_d(v)}.$$

Note that $A_{d-1}(v)$ is modified by a fractional amount in the interval $(-0.5, 0.5)$. Since $A_{d-1}(v)$ is integer valued, it is easy to see that $A'_{d-1}(v) > A'_{d-1}(w)$ if and only if $(A_{d-1}(v), A_d(v)) \succ (A_{d-1}(w), A_d(w))$. We can use this process iteratively to transform any finite dimensional lexicographic labeling into a rational topological labeling. \square

In the rest of this section, we prove that Condition 1 implies Condition 2.

Let N represent the node-arc incidence matrix of the generator graph. Let C be a $d \times m$ matrix. The j -th column of the matrix contains the d -dimensional weight on arc j . The dimension of the weight on the arcs is termed the *weight dimension* of the problem. Let e denote a vector of ones. The dimension of the vector will be clear from the context. We define *cycle arcs* to be the arcs of the periodic graph that lie in a cycle. We define a *multicycle* in a graph to be a collection of directed cycles. We call a multicycle connected if the union of cycles forms a connected graph. Note that if the periodic graph is acyclic, then the set of cycle arcs is empty. The next lemma makes the connection between the cycle arcs in G^∞ and multicycles in G .

Lemma 6 *An arc of G^∞ is a cycle arc if and only if the corresponding arc in the generator graph G is in some connected multicycle of zero weight.*

Definition 8 *We call the arcs of the generator graph G that lie in some multicycle of zero weight the zero multicycle arcs.*

The zero multicycle arcs do not necessarily lie on a connected zero weight multicycle. By Lemma 6, these will determine the cycle arcs of G^∞ . A key step in our algorithm is to determine the set of zero multicycle arcs. More formally, let us define the set S of flows

$$S = \{x : Nx = 0, Cx = 0, x_i \geq 0\}$$

and an index set,

$$I = \{i : \exists x \in S, x_i > 0\}.$$

Thus I is the set of arcs of G that lie on some zero multicycle. Note that the integral solutions

to S correspond to zero multicycles, not necessarily connected. This is true because the constraints $Nx = 0$ restrict x to be a circulation. Therefore, x can be decomposed into a sum of flows around cycles. The second set of constraints restricts the weighted sum of these cycles to be zero. Hence, we have a set of cycles whose weighted sum is zero, all of whose arcs have positive flows on them. We may restrict attention to integral flows since any fractional solution to the system can be scaled to be integral.

Consider the graph $G'(V, I, C)$, which subgraph of the original generator graph restricted to the arcs in I .

Lemma 7 *Let I be the index set of the zero multicycle arcs. If $G' = G'(V, I, C)$ is connected, then G'^{∞} is the set of cycle arcs of G^{∞} .*

Proof:

Follows directly from Lemma 6. □

Therefore, if G' has only one connected component, i.e., G^{∞} is cyclic, there does not exist any topological labeling, (Lemma 6) and we are done. If not, we look at each connected component of G' and apply this algorithm recursively.

The algorithm for developing a lexicographic topological labeling proceeds in the same spirit as those of [1] and [6], in that it may be viewed as developing the solution in a tree-like manner. In order to develop our general procedure, we first treat what may appear to be a very special case of the topological labeling algorithm. Subsequently, we will show how to develop a lexicographic topological labeling by solving this special case in a recursive manner. We consider the following special case of the topological labeling problem.

INPUT

A connected generator graph $G = (V, E, C)$ for an acyclic periodic graph, and a weak lexicographic topological labeling A such that each node of the periodic graph is A -weak.

OUTPUT

A refinement A' of A that is also weakly topological and such that the generator graph $G' = (V, E', C)$ is disconnected, where arc $(v, w) \in E'$ if and only if there exists some copy $(v^k, w^l) \in G^{\infty}$ of arc (v, w) such that $A'(v^k) = A'(w^l)$.

If we have an algorithm to solve the above problem, then we can apply the algorithm recursively to each connected component of the generator graph until we discover a connected zero weight multicycle or each node is a singleton. Note that at the end of this process each node will have a vector label. We can append zeros to the end of these vectors in order to make all the vector labels the same dimension. This vector will be a topological labeling for the nodes of the infinite graph. The recursive computation can be represented in the form of a computation tree. At each node of the computation tree we have an input graph of the type outlined above. The output will be one or more connected components and the same algorithm will be executed at each of the child nodes in the computation tree

to each one of the connected components.

Assume that we have an A -weak lexicographic labeling of a periodic graphs whose generator graph is connected. In order to find a refinement A' of A satisfying the conditions of the output, we first define the following linear program that identifies the zero multicyle arcs in the generator graph.

LPZMA:

$$\begin{aligned} \max \quad & \sum_{(i,j) \in E} y_{ij} \\ Nx &= 0 \\ Cx &= 0 \\ y &\leq x \\ 0 &\leq y \leq 1 \\ x &\geq 0 \end{aligned}$$

Lemma 8 *Let (y^*, x^*) be any optimal solution to LPZMA. Then $y_{ij}^* = 1$ if and only if (i, j) is a zero multicyle arc.*

Proof:

Note that in any optimal solution $y_{ij}^* = \max(1, x_{ij}^*)$. Suppose that (i, j) is a zero-multicycle arc. Then there is a solution x' with $x'_{ij} > 1$ such that $Nx' = 0$ and $Cx' = 0$. Therefore $(y^*, x^* + x')$ is feasible to the LPZMA and therefore $y_{ij}^* = 1$. similarly, if (i, j) is not a zero multicyle arc, then $x_{ij}^* = 0$ and $y_{ij}^* = 0$. \square

The refinement of A into A' described in the procedure, can be based on dual information from LPZMA. However, to make the use of the dual information more direct, we consider a transformation of variables. Substituting $\tilde{x} = x - y$ and $\tilde{y} = y$ we get an alternative formulation MLPZ that also determines the zero multicyle arcs in G .

MLPZ:

$$\begin{aligned} \max \quad & \sum_{(i,j) \in E} \tilde{y}_{ij} \\ N(\tilde{x} + \tilde{y}) &= 0 & (1) \\ C(\tilde{x} + \tilde{y}) &= 0 & (2) \\ \tilde{y} &\leq 1 & (3) \\ \tilde{x}, \tilde{y} &\geq 0 & (4) \end{aligned}$$

Let us associate dual multipliers $\pi = (\pi_i)$ with equations (1), $\gamma = (\gamma_i)$ with equations (2), and $\sigma = (\sigma_{ij})$ with equations (3). The dual to the primal problem MLPZ is shown

below:

MLPD:

$$\begin{aligned} \min \quad & \sum_{(ij) \in E} \sigma_{ij} \\ \pi_i - \pi_j + \gamma C_{ij} & \geq 0 \quad \forall (ij) \in E & (5) \\ \pi_i - \pi_j + \gamma C_{ij} + \sigma_{ij} & \geq 1 \quad \forall (ij) \in E & (6) \\ \sigma_{ij} & \geq 0 \quad \forall (ij) \in E & (7) \end{aligned}$$

It can be shown that the optimal y^* solution to MLPZ is unique. We use the dual variables associated with each linear program to generate the lexicographic labels. We will first present a lemma that characterizes the optimal solution to the linear program MLPZ.

Lemma 9 *Let (x^*, y^*) and $(\pi^*, \gamma^*, \sigma^*)$ denote a primal and a dual optimal solution to MLPZ and MLPD respectively. Consider the arc (v, w) in G and let (v^k, w^l) where $l - k = C_{uv}$, be some copy of the arc in G^∞ . If $y_{uv}^* = 1$, then,*

$$\gamma k - \pi_v^* = \gamma l - \pi_w^*.$$

If $y_{uv}^* = 0$,

$$\gamma k - \pi_v^* < \gamma l - \pi_w^*.$$

Proof:

Consider the linear program MLPZ and the dual MLPD. Let (x^*, y^*) and $(\pi^*, \gamma^*, \sigma^*)$ denote a primal and a dual optimal solution respectively. From equations (5) and (6) note that $\sigma_{ij} = 1$ is dual feasible. Since the objective function of the dual is to minimize $\sum_{(ij) \in E'} \sigma_{ij}$ the value of σ_{ij}^* will never be strictly bigger than one in the optimal solution. By complementary slackness we have,

$$y_{ij}^* = 0 \Rightarrow \sigma_{ij}^* = 0$$

We now show that

$$y_{ij}^* = 1 \Rightarrow \sigma_{ij}^* = 1.$$

If for some (i, j) , $y_{ij}^* = 1$ and $\sigma_{ij}^* < 1$ then because the objective function of the primal and the dual problems are equal there exists some (i', j') such that $y_{i'j'}^* = 1$ and $\sigma_{i'j'}^* > 1$. This is contradicts the fact that $\sigma_{ij}^* \leq 1$. Therefore,

$$y_{ij}^* = \sigma_{ij}^*.$$

We now consider the other complementary slackness condition.

$$y_{ij}^* (\pi_i^* - \pi_j^* + \gamma^* C'_{ij} + \sigma_{ij}^* - 1) = 0$$

Therefore,

$$y_{ij}^* = 1 \Rightarrow \pi_i^* - \pi_j^* + \gamma^* C'_{ij} + \sigma_{ij}^* = 1$$

and since $y_{ij}^* = 1$ implies that $\sigma_{ij}^* = 1$, and by (2)

$$y_{ij}^* = 1 \Rightarrow \pi_i^* - \pi_j^* + \gamma^* C'_{ij} = 0$$

Also $y_{ij}^* = 0$ implies that $\sigma_{ij}^* = 0$ and by (9)

$$y_{ij}^* = 0 \Rightarrow \pi_i^* - \pi_j^* + \gamma^* C'_{ij} > 0$$

Consider two nodes in G^∞ , say v_k and w_l . There is an arc between these two nodes in G^∞ if and only if there is an arc in G between v and w with weight $(l - k)$. If this arc is in the solution to the primal problem, i.e., $y_{ij}^* = 1$ for this arc, then (10) may be restated as

$$\pi_v^* - \pi_w^* + \gamma^*(l - k) = 0,$$

and therefore,

$$\gamma^*k - \pi_v^* = \gamma^*l - \pi_w^*.$$

If $y_{vw}^* = 0$ then

$$\pi_v^* - \pi_w^* + \gamma^*(l - k) > 0$$

or

$$\gamma^*k - \pi_v^* < \gamma^*l - \pi_w^*.$$

Since π and γ are rational vectors, all the labels can be scaled to be integral. \square

Let $A(v^k)$ be the d dimensional lable of node $v^k \in G^\infty$. We denote component i of this vector by $[A(v^k)]_i$. The first d dimensions of the refinement $A'(v^k)$ will be the same as $A(v^k)$. The $d + 1$ dimension of $A'(v^k)$ is given by $\gamma^*k - \pi_v^*$. Note that $y_{uv}^* = 1$ implies that (u, v) is a zero multicycle arc. By Lemma 6, note that the arc in the generator graph corresponding to the set of zero multicycle arcs cannot be connected if G^∞ is acyclic. Also note that all the non-zero multicycle arcs are A' strong. Therefore the only A' weak arcs are the zero multicycle arcs.

For the next lemma assume that all the arcs are initially given a label of zero. Note that this is a weak lexicographic labeling of the graph.

Lemma 10 *All arcs in G^∞ are weakly lexicographically labelled initially. Consider an arc (u, v) in G . All copies of this arc in G^∞ remain weakly lexicographically labeled while (u, v) is in the set of some zero multicycle arcs. All copies of (u, v) are strongly lexicographically labelled at the tree node at which it is determined that (u, v) is not a zero multicycle arc, and they remain strongly lexicographically labeled thereafter.*

Proof:

Consider some arc (u, v) in G with weight C_{uv} . We want to show that any copy of this arc in G^∞ is weakly lexicographically labelled until it is determined that (u, v) is shown not as a zero multicycle arc. Let (u^p, v^q) be a copy of (u, v) and thus $q - p = C_{uv}$. Let i^* be the node in the computation tree whose linear program demonstrates that (u, v) is not a zero multicycle arc. Let P be the path in the computation tree from the root node to i^* . Let the length of the path be l , i.e., i^* is at level l in the computation tree. At the initialization step all nodes are given the label zero. Therefore, all copies of arc (u, v) are initially weakly lexicographically labeled. If the length of the path is zero, i.e., i^* is the root node arc (u, v) is not in the zero multicycle arcs at the root node then by lemma 9,

$$[A(u^p)]_1 < [A(v^q)]_1$$

and the arc is strongly lexicographically labeled at this stage and will remain strongly lexicographically labeled for the rest of the algorithm. If the arc is a zero multicycle arc, then

$$[A(u^p)]_1 = [A(v^q)]_1,$$

and the arc remains weakly lexicographically labeled. This will be true for the tree nodes upto depth $l - 1$ along P . At node i^* we solve an instance of MLPZ and determine that $y_{uv}^* = 0$, i.e., (u, v) is not a zero multicycle arc. By lemma 9,

$$[A(u^p)]_i = [A(v^q)]_i \quad i = 1, 2, \dots, l - 1,$$

and

$$[A(u^p)]_l < [A(v^q)]_l.$$

Therefore the arc gets strongly lexicographically labelled at this node. The arc remains strongly lexicographically labelled since the algorithm modifies only the components greater than l during the rest of the algorithm. □

Since the graph is acyclic, no arc in G is a zero multicycle arcs throughout the computations and therefore all the arcs in G^∞ get strongly lexicographically labelled at the end of the algorithm.

An Example:

Consider the one dimensional generator graph shown in Figure 3 and the corresponding periodic graph in Figure 4.

Initially all nodes are given a label of zero. The optimal solution to the MLPZ at the root of the computation tree is 2. The optimal solution consists of the the loop arc on nodes 1 and 2. The second component of the labeling vector is generated using the optimal dual multipliers. Since arc $(1, 2)$ is not part of the zero multicycle at this node it will be labeled strongly at this node of the computational tree. The two nodes at the next level involve solving MLPZ for each of the connected components of the zero multicycle arcs. The first connected component is the loop at node 1 and the second connected component is the loop

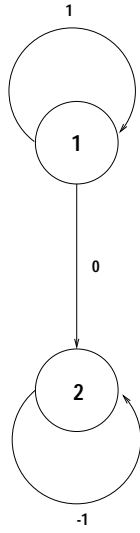


Figure 3: One Dimensional Generator Graph

at node 2. The optimal solution to each of these LPs is zero and the optimal dual solution is used to generate the third component of the label. Let $A(1^k)$ represent the label for node 1 at lattice point k and $A(2^k)$ represent the label for node 2 at lattice point k . Then

$$A(1^k) = \begin{pmatrix} 0 \\ -1 \\ k \end{pmatrix} \quad A(2^k) = \begin{pmatrix} 0 \\ 0 \\ -k \end{pmatrix}$$

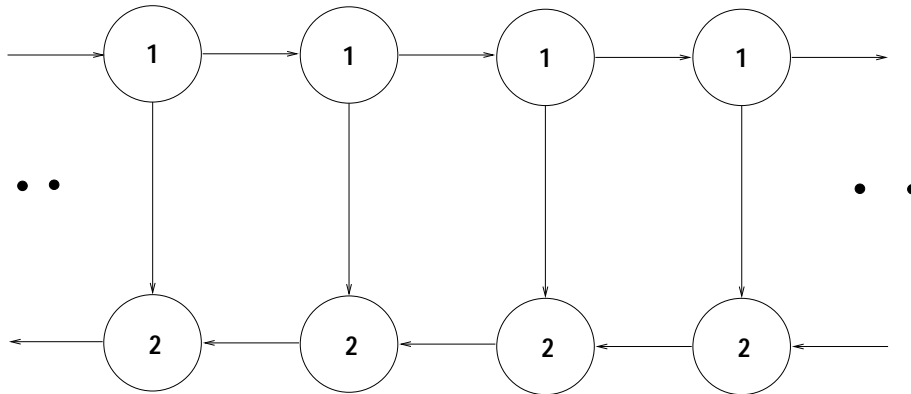


Figure 4: One Dimensional Periodic Graph

The rational topological labels for the nodes in G^∞ are computed using the result of Lemma 5. The rational labels for the nodes are

$$A(1^k) = \begin{cases} \frac{-1-k}{1+2k} & \text{if } k \geq 0 \\ \frac{-1+3k}{1-2k} & \text{if } k < 0 \end{cases}$$

$$A(2^k) = \begin{cases} \frac{-k}{1+2k} & \text{if } k \geq 0 \\ \frac{-k}{1-2k} & \text{if } k < 0 \end{cases}$$

The labels are shown graphically in Figure 5.

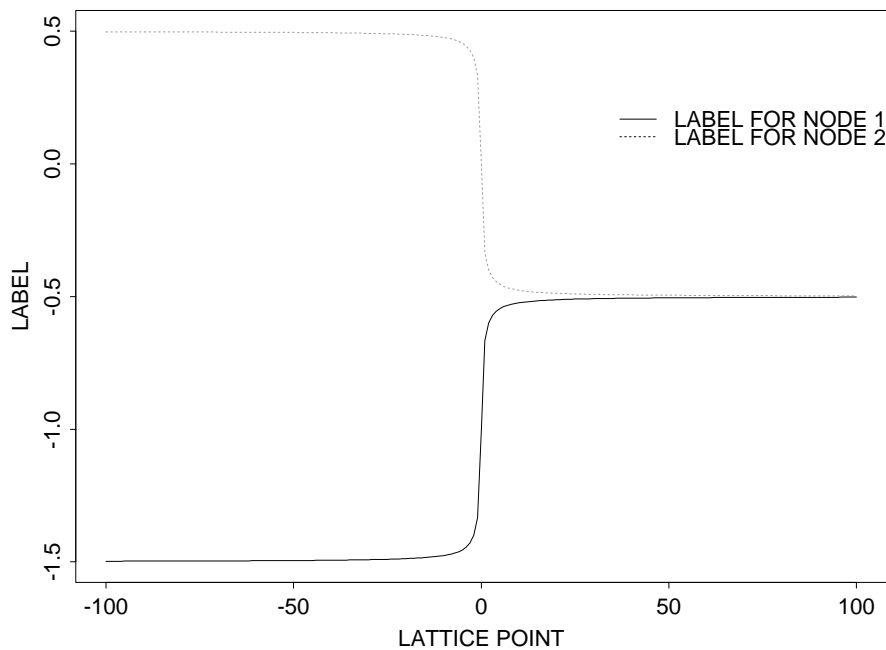


Figure 5: Plot of the Rational Labels for the Two Nodes

In the next section we consider the cases where if G^∞ is acyclic, then it can be topologically labeled with integers and we give sufficient conditions for the existence of an integral topological labeling.

3 Integral Topological Labeling

In this section we provide a sufficient condition for the existence of an integer topological labeling, and a sufficient condition for its non-existence.

Lemma 11 *If there is no zero multicycle in G , then the nodes in G^∞ has an integral topologically labeling.*

Proof:

Let us assume that the optimal solution to the primal-dual pair is 0. This certainly implies that that G^∞ is acyclic (but it is only a sufficient condition). In this case, $y_{uv}^* = 0 \ \forall (u, v) \in E$. Therefore if we give node v^k a label of

$$A(v^k) = \gamma^* k - \pi_v^*$$

then this defines a topological labeling of the nodes. Recall that we can assume that the labels are all integral. Therefore there exist integer labels such that for each arc $(u^p, v^q) \in G^\infty$ we have

$$A(u^p) < A(v^q).$$

□

Hence, in the special class of periodic graphs in which the optimal solution to MLPZ at the root node of the computation tree is zero, it is possible to topologically label the nodes with integers. We shall call this class of periodic graphs *strongly acyclic*. For example, if the weight of the loop arc at node 2 is changed from -1 to 1 then the solution for MLPZ at the root node of the computation tree is zero. In this case (ignoring the initial label of zero given to all nodes), there exists an integral topological label for the nodes in G^∞ where $A(1^k) = k - 1$ and $A(2^k) = k$.

The converse of the Lemma 11 is not true as is illustrated by the generator graph in Figure 6.

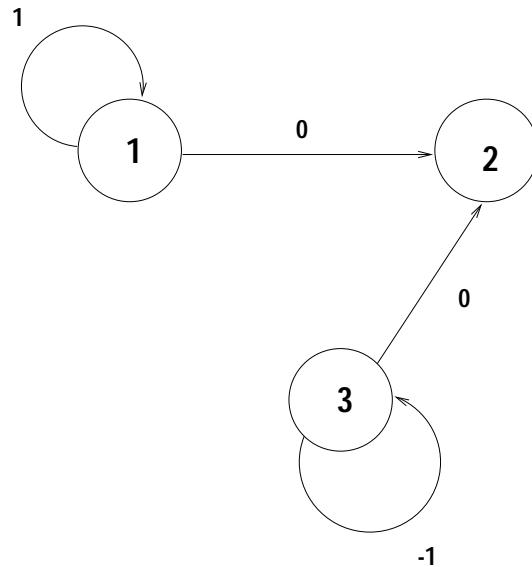


Figure 6: Generator Graph

The generator graph is connected and has a zero-multicycle consisting of the loop arcs at node 1 and node 3. This implies that the solution of MLPZ at the root node is two. However, there exists an integral topological labeling of G^∞ .

$$\begin{aligned} A(1^p) &= p & \forall p \in Z. \\ A(2^p) &= |p| + 1 & \forall p \in Z. \\ A(3^p) &= -p & \forall p \in Z. \end{aligned}$$

In the rest of this section we will establish a sufficient condition for the non-existence of an integral topological labeling of an acyclic periodic graph.

Definition 9 *The maximum path length between two nodes u^p and v^q in G^∞ is defined as the number of arcs in the longest path between u^p and v^q . This is represented as $\lambda(u^p, v^q)$.*

In the next lemma we will show that the length of the paths between any two nodes in a strongly acyclic graph is bounded.

Lemma 12 *If there is an integer topological labelling of the nodes in G^∞ then*

$$\lambda(u^p, v^q) < \infty \quad \forall u^p, v^q.$$

Proof:

Since any path can have only increasing labels,

$$\lambda(u^p, v^q) \leq A(v^q) - A(u^p).$$

And since we have an integer topological labelling of the nodes,

$$\lambda(u^p, v^q) < \infty \quad \forall u^p, v^q.$$

□

In the next lemma we will give a partial converse to the above statement.

Lemma 13 *Suppose G is strongly connected. If the graph is not strongly acyclic, then*

$$\lambda(u^p, v^q) = \infty \text{ for some } u^p, v^q.$$

Proof:

The graph is not strongly acyclic implies that the solution to MLPZ at the root node is non zero. The set of zero multicycle arcs S are not empty. If the arcs in S is connected then there is a cycle in G^∞ and it is not possible to label the nodes topologically. Let us now assume that the set S is not connected. Let the connected components of S be $\{S_1, S_2, \dots, S_t\}$. Since the graph is strongly connected there is a directed path P_i between S_i and S_{i+1} for $i = 1, 2, \dots, t-1$. Let $P = \cup_{i=1}^{t-1} P_i$. The $w(P)$ be the weight of path P .

Consider some node v in S_1 and w in S_t . Consider paths between v^p and $w^{p+w(P)}$. We know that there is a path between these two nodes using all arcs in S and P exactly once. Therefore there is a path of length $|S| + t - 1$.

$$\lambda(v^p, w^{p+w(P)}) \geq |S| + t - 1.$$

Since S is the set of zero multicycle arcs, we can take M copies of the arcs in S along with one copy of the arcs in P to get a path from v^p to $w^{p+w(P)}$. The number of arcs in this path is $M|S| + t - 1$. Therefore,

$$\lambda(v^p, w^{p+w(P)}) \geq M|S| + t - 1.$$

Since this is true for all M ,

$$\lambda(v^p, w^{p+w(P)}) = \infty.$$

□

Therefore if G is strongly connected, then G^∞ has an integral topological labelling if and only if G is strongly acyclic.

4 Conclusion

In this paper, we extend the definition of topological labeling of finite directed acyclic graphs to infinite acyclic periodic graphs. We show that exist rational topotlogical labels for these infinite acyclic periodic graphs. We also show how to obtain these labels by solving a sequence of linear programs. We also give sufficient conditions for the existence of an integral topological labeling for infinite acyclic periodic graphs.

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